Algebraic integrability of confluent Neumann system

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Abstract. In this paper we study the Neumann system, which describes the harmonic oscillator (of arbitrary dimension) constrained to the sphere. In particular we will consider the confluent case where two eigenvalues of the potential coincide, which implies that the system has S^1 symmetry. We will prove complete algebraic integrability of confluent Neumann system and show that its flow can be linearized on the generalized Jacobian torus of some singular algebraic curve. The symplectic reduction of S^1 action will be described and we will show that the general Rosochatius system is a symplectic quotient of the confluent Neumann system, where all the eigenvalues of the potential are double.

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1. Introduction

Neumann system describes motion of a particle constrained to the n-dimensional sphere S^n under quadratic potential. The potential is given in ambient coordinates $q = (q_1, \ldots, q_{n+1}) \in \mathbb{R}^{n+1}$ by the potential matrix $A = \operatorname{diag}(a_1, \ldots, a_{n+1})$ as

$$V(q) = \frac{1}{2}\langle Aq, q\rangle = \frac{1}{2}\sum_{i=1}^{n+1} a_i q_i^2.$$

In the generic case where all the eigenvalues of the potential a_i are different, the Neumann system is algebraically completely integrable and its flow can be linearized on the Jacobian torus of an algebraic spectral curve [1, 2, 3]. A standard approach to study integrable systems is by writing down the system in the form of Lax equation, which describes the flow of matrices or matrix polynomials with constant eigenvalues, i.e. the isospectral flow [2, 4]. Eigenvalues of the isospectral flow are the first integrals of the integrable system and Lax representation maps Arnold-Liouville tori into the real part of the isospectral manifold consisting of matrices with the same spectrum. A quotient of the isospectral manifold by a suitable gauge group is in turn isomorphic to the open subset of the Jacobian of the spectral curve [5].

Two different Lax equations are known for a generic Neumann system with n degrees of freedom: one is using $(n + 1) \times (n + 1)$ matrix polynomials of degree 2 [2] and the other is using 2×2 matrix polynomials of degree n [3, 5]. The $(n+1)\times(n+1)$ Lax equation was used by Audin [6] to describe the Arnold-Liouville tori for geodesic motion on an ellipsoid, which is equivalent to the Neumann system.

In contrast to the generic Neumann system, the special confluent case in which some eigenvalues of the potential coincide has not received much attention. In this paper we will consider the confluent case with two of the eigenvalues being the same and the Neumann system having an additional S^1 rotational symmetry. We will show that the confluent Neumann system is algebraically completely integrable and that its flow can be linearized on the generalized Jacobian of a singular algebraic curve. We will describe the symplectic reduction of the S^1 action, which will yield an alternative description of the Rosochatius system [7, 8, 9] as a symplectic quotient of the confluent Neumann system. Mechanically speaking, Rosochatius system can be seen as a Neumann system on a rotating sphere. More generally one can describe general Rosochatius system as a reduction of confluent Neumann system with all eigenvalues of the potential being double [1, 8]. Combined with the proof of integrability of the confluent Neumann system this result will also give an alternative proof of the algebraic integrability of the Rosochatius system.

When applying the 2×2 Lax equation to the confluent case, the resulting Lax equation in fact describes the Rosochatius system and not the confluent Neumann system [10, 11, 12]. We will therefore use $(n + 1) \times (n + 1)$ Lax equation, where the resulting spectral curve is singular in the confluent case. Following the standard procedure and normalizing the spectral curve results in the loss of one degree of freedom. In order to avoid that, we will use the generalized Jacobian of the singular spectral curve to linearize the flow as in [13, 14]. The generalized Jacobian is an extension of the "ordinary" Jacobian by a commutative algebraic group (see [15] for more detailed description). In our case the extension will be the group $\mathbb{C}^*$ that corresponds to the rotational symmetry of the initial system. Generalized Jacobian was used by others to linearize the flow of other integrable systems with rotational symmetry for example spherical pendulum or Lagrange top [16, 14, 17, 13]. Our study however will give a more detailed description of the relation between symplectic reduction and algebraic reduction from generalized to ordinary Jacobian.

One should also mention that apart from classical case there has been a lot of interest in quantum case [18, 19, 20] for both Neumann and Rosochatius system.

After the introduction, Hamiltonian reduction and Liouville integrability of the confluent Neumann system are discussed in section 2. In section 3 we study $(n+1)\times(n+1)$ Lax equation and corresponding isospectral manifolds. Our main result is formulated in theorem 3.5 and describes the relation of Arnold-Liouville tori to the generalized Jacobian of the singular spectral curve. As a corollary the complete algebraic integrability and Liouville integrability of the confluent Neumann system will follow. We conclude with proposition 3.8, which describes the bifurcation diagram of the energy momentum map in terms of algebraic data.

2. Hamiltonian description

The Neumann system describes a particle on a sphere (of arbitrary dimension) under the influence of quadratic potential. We can write it as a Hamiltonian system on the cotangent bundle of the sphere T^*S^n with canonical symplectic form ω_c and the Hamiltonian H given in ambient coordinates $(q,p)\in\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ as

$$H(q,p)=\frac{1}{2}\left(\|p\|^2+\langle q,Aq\rangle\right).$$

The potential is given by a positive definite linear operator A on $\mathbb{R}^{n+1}$. For simplicity we will assume that A is diagonal with positive eigenvalues a_i. A consequence of positivity is that the Hamiltonian is proper and the energy level sets - and hence Arnold-Lioville tori - are compact. The equations of motion in Hamiltonian form are

$$\begin{aligned}\dot q &= p \\ \dot p &= -Aq+\varepsilon q\end{aligned} \tag{1}$$

where $\varepsilon=\|p\|^2+c$ is chosen so that $\|q\|=1$ and the particle stays on the sphere.

2.1. Reduction of the symmetry

Let us consider the confluent case where all the eigenvalues a_i of the potential A are distinct, except $a_n=a_{n+1}$. The action φ' of $S^1=SO(2,\mathbb{R})$ on S^n, given by rotations in the q_n,q_{n+1} plane, leave the potential invariant and can be lifted to the symplectic action

$$\varphi: T^*S^n\times S^1\to T^*S^n$$

on the cotangent bundle that leaves the Hamiltonian H invariant. We would like to reduce this action and describe the resulting reduced system in more detail.

Let K be the moment map for the lifted action φ. The map K is a real function on T^*S^n as $\mathfrak{so}(2)^*\simeq\mathbb{R}$ and is the *angular momentum* for the rotations in q_n,q_{n+1} plane

$$K(q,p)=q_np_{n+1}-q_{n+1}p_n.$$

Since the action φ is not free, the description of reduced system is more complicated. Let us denote with F the set of points on the sphere, where the action φ' is not free. This is precisely the fixed point set given by the condition

$$r^2 = q_n^2 + q_{n+1}^2 = 0$$

The set F is a codimension-2 great sphere on S^n. Note that on F, the value k of the moment map K equals 0. The set of regular points $S^n_{reg} = S^n - F$ is an open subset of S^n on which the action is free.

The reduced system is defined on symplectic quotients (M_k, ω_k), which are parametrized by the value k of the moment map K. We will use the operator $//_k$ to denote the symplectic quotient. By definition

$$M_k = T^*S^n //_k S^1 := K^{-1}(k)/S^1$$

and ω_k is defined with $\pi^*\omega_k = \omega_c|_{K^{-1}(k)}$, where π is the quotient projection. For $k \neq 0$ the fiber $K^{-1}(k)$ is a corank-1 sub-bundle of the cotangent bundle over the set of regular points $T^*S^n_{reg}$. Since the action is free on $K^{-1}(k)$, the standard result for lifted actions gives

$$(M_k, \omega_k) = (T^*(S^n_{reg}/S^1), \omega_c + \omega_p),$$

where ω_p is the magnetic term coming from the curvature of the mechanical connection. In our case the mechanical connection is flat and $\omega_p = 0$ [21]. The quotient manifold S^n_{reg}/S^1 is a dimension-$(n-1)$ open half sphere S^{n-1}_+ as we can see if we introduce cylindrical coordinates $(q_1, \dots, q_{n-1}, r(\cos\varphi, \sin\varphi))$. The set S^n_{reg} is given by the condition $r \neq 0$ and the quotient S^n_{reg}/S^1 can be parametrized by $(q_1, \dots, q_{n-1}, r) \in S^{n-1}$ for $r > 0$. We can also see directly by using cylindrical coordinates that the perturbation ω_p of the canonical symplectic form is zero.

For $k = 0$ the description of M_0 is more complicated, since the action on $K^{-1}(0)$ is not free, and M_0 is not a manifold. We will remedy this by considering a singular double cover of the reduced phase space instead.

The set $K^{-1}(0)$ is not a sub-bundle of T^*S^n, because at fixed points $r = 0$ the fiber has full rank and over S^n_{reg} the fiber has codimension 1. However, we can still pass the quotient on the base manifold because the action is a cotangent lift. The quotient space S^n/S^1 is a dimension-$(n-1)$ closed half sphere $(S^{n-1}_+)^c$ and can be parametrized by cylindrical coordinates $(q_1, \dots, q_{n-1}, r) \in S^{n-1}$, with $r \geq 0$. The symplectic quotient M_0 is a cotangent bundle over S^n/S^1, in the sense that the fiber at a singular point with $r = 0$ is a closed half plane of "positive differentials" (the differential, which is positive at the directions normal to the boundary of S^n/S^1). In that sense we can write

$$(M_0, \omega_0) = (T^*(S^{n-1}_+)^c, \omega_c).$$

To conclude the construction of the reduced system we have to calculate the reduced Hamiltonian. This can be easily done using the parametrization with cylindrical coordinates $(\hat{q}, \hat{p}) = ((q_1, \dots, q_{n-1}, r), (p_1, \dots, p_{n-1}, p_r))$

$$\begin{aligned}
H(q,p) &= \frac{1}{2}\left(\sum_{i=1}^{n-1}\left(p_i^2 + a_i q_i^2\right) + p_n^2 + p_{n+1}^2 + a_n(q_n^2 + q_{n+1}^2)\right) = \\
&= \frac{1}{2}\left(\sum_{i=1}^{n-1} p_i^2 + p_r^2 + \frac{k^2}{r^2} + \sum_{i=1}^{n-1} a_i q_i^2 + a_n r^2\right) = \\
&= \frac{1}{2}\left(\|\hat{p}\|^2 + V_k(\hat{q}) + \langle \hat{q}, \hat{A}\hat{q}\rangle\right) = H_r(\hat{q}, \hat{p}),
\end{aligned}$$

where $\hat{A} = \mathrm{diag}(a_1, \dots, a_n)$. The additional term $V_k(\hat{q}) = \frac{k^2}{r^2}$ corresponds to the centrifugal system force, induced by the rotation.

To simplify the description of the reduced system, we will use the whole sphere S^{n-1} instead of only half of it. The half sphere $(S_+^{n-1})^c$ can be also viewed as a quotient of S^{n-1} by the group $\mathbb{Z}_2$ acting with reflections $r \mapsto -r$. The reduced Hamiltonian $H_r(\hat{q}, \hat{p})$ can be lifted on (T^*S^{n-1}, ω_c), since it only depends on r^2 and p_r^2 and is invariant for $\mathbb{Z}_2$ action. We have thus constructed a Hamiltonian system $(T^*S^{n-1}, \omega_c, H_r)$ such that its reduction coincides with the reduced Neumann system.

Theorem 2.1. *The singular symplectic quotient (M_k, ω_k, H_r) of the Neumann Hamiltonian system $(T^*S^{n-1}, \omega_c, H_A)$ with potential given by the matrix $A = diag(a_1, \dots, a_n, a_n)$ is isomorphic to the quotient by the $\mathbb{Z}_2$ action of the perturbed Neumann system $(T^*S^{n-1}, \omega_c, H_{\hat{A}} + V_k)$ with potential matrix $\hat{A} = \mathrm{diag}(a_1, \dots, a_n)$.*

The symplectic quotient of confluent Neumann system is in fact a special case of Rosochatius system, which was studied in [7, 8, 9, 10]. A general Rosochatius system is a Hamiltonian system on T^*S^n with potential

$$V = \frac{1}{2}\left(\sum_{i=1}^{n+1} a_i q_i^2 + \frac{k_i^2}{q_i^2}\right). \tag{2}$$

We can generalize the procedure from the above to the case where potential matrix has all the eigenvalues double $A = \mathrm{diag}(a_1, a_1, a_2, a_2, \dots, a_{n+1}, a_{n+1})$. This gives a mechanical interpretation of Rosochatius system as Neumann system on a rotating sphere.

Corollary 2.2. *The symplectic quotient of the confluent Neumann system with potential matrix $A = \mathrm{diag}(a_1, a_1, a_2, a_2, \dots, a_{n+1}, a_{n+1})$ by the group of rotations $(S^1)^{n+1}$ is isomorphic to the quotient by the group of reflections $\mathbb{Z}_2^{n+1}$ of the Rosochatius system on T^*S^n given by the potential (2). The coefficients k_i of the rational part are the values of the angular momentum of rotations in corresponding eigenplanes of the potential matrix A.*

This result was already mentioned in [1] and was used in [8] to study the Rosochatius potential without polynomial part.

2.2. Integrability

Hamiltonian system with Hamiltonian H on the symplectic manifold of dimension $2n$ is called *integrable* if there exist n functionally independent pairwise Poisson commutative first integrals, one of which is H. To show that the system is *completely algebraically integrable,* we will prove that the level sets of the first integrals are real parts of the extensions of Abelian varieties by $\mathbb{C}^*$ (see [3] for a definition of complete algebraic integrability).

The integrability of the confluent Neumann system is a consequence of the integrability of the generic Neumann system, which is well known[1, 22]. The first integrals for the confluent case can be obtained by taking the limit $a_n \to a_{n+1}$ on Uhlenbeck's integrals for generic Neumann system:

$$F_i^g = q_i^2 + \sum_{j \neq i} \frac{(q_i p_j - q_j p_i)^2}{a_j - a_i}. \tag{3}$$

A set of commuting integrals, equivalent to (3), tends to a set of commuting integrals for the confluent case

$$\begin{aligned}&\{F_1^g,\dots,F_{n-1}^g,F_n^g+F_{n+1}^g,\frac{1}{2}(a_n-a_{n+1})(F_n^g-F_{n+1}^g)\}\to\\&\to\{F_1,\dots,F_{n-1},F_n,K^2\}\end{aligned}$$

when taking the limit $a_n\to a_{n+1}$. The set of commuting integrals for the confluent Neumann system is given by

$$\begin{aligned}F_i\;&=q_i^2+\sum_{j\neq i}\frac{(q_ip_j-q_jp_i)^2}{a_j-a_i};\quad i<n\\F_n\;&=q_n^2+q_{n+1}^2+\sum_{j<n}\frac{(q_jp_n-q_np_j)^2+(q_jp_{n+1}-q_{n+1}p_j)^2}{a_j-a_n}\\K^2&=(q_np_{n+1}-q_{n+1}p_n)^2.\end{aligned}\tag{4}$$

Note that K is angular momentum for the rotations in q_n,q_{n+1} plane. We can also verify that the integrals in (4) are not independent but satisfy the same relations as Uhlenbeck's integrals in the generic case

$$\sum_{i=1}^n F_i=1$$

and that the Hamiltonian H can be expressed as a linear combination of F_i and K^2

$$\sum_{i=1}^n a_iF_i+K^2=2H.$$

The Poisson brackets of F_i are continuous functions of a_i so the commutativity of the integrals is preserved when taking the limit $a_n\to a_{n+1}$. Commutativity of K with F_i also follows from the fact that F_i are invariant for S^1 action, generated by K. To conclude the proof of integrability one needs to verify the independence of the integrals (4) (up to relation $\sum F_i=1$). The commutativity of the integrals F_i and K^2 also follows from the AKS theorem, if we write F_i an K^2 as invariant functions of Lax matrix on appropriate loop algebra. This is a standard way to prove commutativity of Uhlenbeck's integrals for the generic Neumann case [10, 5].

Let us combine all the first integrals in a map

$$\begin{aligned}F_{EM}&:T^*S^n\to\mathbb{R}^n\\&(q,p)\mapsto(F_1,\dots,F_{n-1},K)\end{aligned}$$

we will call *energy momentum map.* The fundamental property of integrable systems is that the level sets of energy momentum map F_{EM} are n-dimensional tori, on which the flow can be linearized. We will also use the complexified version of F_{EM}, defined on $(T^*S^n)^{\mathbb{C}}$.

Theorem 2.3. *The confluent Neumann system (T^*S^n,ω_c,H) is algebraically completely integrable system.*

We have seen that by taking the limit $a_n\to a_{n+1}$, $\{F_1,\dots,F_{n-1},K^2\}$ is a set of n commuting first integrals for symmetric Neumann system and we will show later in 3.3 that they are functionally independent. We will also prove the part about algebraically complete integrability in subsection 3.3.

From the fact that the symmetric Neumann system is integrable also follows that its symplectic quotient is integrable. This gives alternative proof of the integrability of Rosochatius system, which is a symplectic quotient of the symmetric Neumann system.

Corollary 2.4. *The Rosochatius system* $(T^*S^n, \omega_c, H + V_r)$ *with* $V_r = \sum \frac{ki^2}{q_i^2}$ *is algebraically completely integrable system.*

3. Lax representation of the confluent Neumann system

In this section we will use Lax equation for $(n+1) \times (n+1)$ matrix polynomials to study confluent Neumann system with n degrees of freedom. We will show that the flow of the system can be linearized on the Jacobian of the singular spectral curve and that the system is completely algebraically integrable. This will also conclude the proof of Liouville integrability and yield the description of bifurcation diagram for energy momentum map.

3.1. Lax equation in $\tilde{\mathfrak{gl}}(n+1, \mathbb{C})$

Let us write the confluent Neumann system (1) as an isospectral flow of matrix polynomials in loop algebra $\tilde{\mathfrak{gl}}(n+1, \mathbb{C})$. We will use the Lax equation introduced by Moser [1]. The *loop algebra* $\tilde{\mathfrak{gl}}(n+1, \mathbb{C})$ consists of Laurent polynomials with coefficients in $\mathfrak{gl}(n+1, \mathbb{C})$, and can be written as a tensor product $\mathfrak{gl}(n+1, \mathbb{C}) \otimes \mathbb{C}[\lambda, \lambda^{-1}]$. Consider complexified Neumann system (1) on a subspace $(T^*S^n)^{\mathbb{C}} \subset \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ defined by constraints $\sum q_i^2 = 1$ and $\sum q_i p_i = 0$. We introduce *Lax matrix polynomial* from $\tilde{\mathfrak{gl}}(n+1, \mathbb{C})$

$$L(\lambda) = A\lambda^2 + q \wedge p\ \lambda - q \otimes q\ ; \quad q, p \in \mathbb{C}^{n+1} \tag{5}$$

where A is the potential matrix of the Neumann system and $(q, p) \in (T^*S^n)^{\mathbb{C}}$. Neumann system either generic or confluent can be written as *Lax equation*

$$\frac{\mathrm{d}}{\mathrm{d}t} L(\lambda) = [M(\lambda), L(\lambda)]. \tag{6}$$

with $M(\lambda) = A\lambda + q \wedge p = (\lambda L(\lambda))_+$ (the subscript $_+$ denotes the polynomial part of an element of $\tilde{\mathfrak{gl}}(n+1, \mathbb{C})$)[2]. The flow of (6) is isospectral as it conserves the spectrum of the matrix $L(\lambda)$. This means that the characteristic polynomial $P(\lambda, \mu)$ and the corresponding affine spectral curve, defined by equation

$$P(\lambda, \mu) = \det(L(\lambda) - \mu) = 0$$

do not change along the flow and can be expressed only with the values of the first integrals of Neumann system. In order to avoid unnecessary singularities at the infinity, the affine curve is completed in the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(2)$ over $\mathbb{P}^1$, which is given by the transition function $(\lambda, \mu) \mapsto (\lambda^{-1}, \lambda^{-2}\mu)$. The completion of the affine spectral curve in $\mathcal{O}_{\mathbb{P}^1}(2)$ is called *the spectral curve* of $L(\lambda)$ and denoted by $\mathcal{C}_m$. Note that for any given $L(\lambda)$ there are also naturally defined a map $\lambda : \mathcal{C}_m \to \mathbb{P}^1$ and a section μ of the bundle $\lambda^*\mathcal{O}_{\mathbb{P}^1}(2)$ apart from the spectral curve $\mathcal{C}_m$.

The spectral curve of $L(\lambda)$ is hyperelliptic and we can see that by introducing new variables $x = \lambda^{-2}\mu$ and $y = \lambda \prod_{i=1}^{n+1}(a_i - x)$, where a_i are the eigenvalues of the potential matrix A. The equation we obtain is:

$$y^2 = Q(x) \prod_{i=1}^{n+1} (a_i - x), \tag{7}$$

where $Q(x)$ is a polynomial of degree n [1, 6]. The coefficients of $Q(x)$ are first integrals of the confluent Neumann system and if we write $Q(x)$ in terms of Lagrange interpolating polynomials over the points $a_1, \ldots, a_n, a_n$ using notation with partial fractions

$$Q(x) = \prod_{j=1}^{n+1}(a_j - x)\left(\frac{F_n}{a_n - x} + \frac{K^2}{(a_n - x)^2} + \sum_{i=1}^{n-1}\frac{F_i}{a_i - x}\right), \tag{8}$$

the coefficients we obtain are the integrals (4) we met before .

In the confluent case, where $a_n = a_{n+1}$, the product $\prod(a_i - x)$ has a quadratic factor $(a_n - x)^2$ and the spectral curve has a singular point S given by $(x, y) = (a_n, 0)$ or $(\lambda, \mu) = (\infty, a_n)$. The singularity S is a double point for $K \neq 0$ and a cusp for $K = 0$. The smoothness of the spectral curve is closely related to the regularity of the matrix $L(\lambda)$. Recall that a matrix $B \in \mathfrak{gl}(r, \mathbb{C})$ is called *regular* if all the eigenspaces of B are one dimensional.

Proposition 3.1. *Let $\mathcal{C}$ be the spectral curve of matrix polynomial $L(\lambda)$ and $a \in \mathbb{P}^1$. If all the points $\lambda^{-1}(a) \in \mathcal{C}$ are smooth, then the matrix $L(a)$ is regular.*

For proof see [5, 13]. The value of $L(\lambda)\lambda^{-2}$ at $\lambda = \infty$ is the matrix A and the singularity at S is a consequence of the fact that A is not regular when $a_n = a_{n+1}$.

Let $\mathcal{C}$ be normalization of the spectral curve $\mathcal{C}_m$, which is described by

$$w^2 = Q(x)\prod_{i=1}^{n-1}(a_i - x), \tag{9}$$

where $w = y/(a_n - x)$. We will call $\mathcal{C}$ the *normalized spectral curve* of $L(\lambda)$. There is a map $\pi : \mathcal{C} \to \mathcal{C}_m$ that is biholomorphic everywhere except at the inverse image of the singular point S. The inverse image $\pi^{-1}(S)$ consists of two points $\{P_+, P_-\}$ for $K \neq 0$ and a point P_0 for $K = 0$. In case when $K \neq 0$, the curve $\mathcal{C}_m$ is obtained from $\mathcal{C}$ by identifying the points $\{P_+, P_-\}$ into the singular point S. The singular curve $\mathcal{C}_m$ can be described as a singularization of $\mathcal{C}$ given by *modulus* m (see [15] for details). The modulus is $m = P_+ + P_-$ for $K \neq 0$ and $m = 2P_0$ for $K = 0$.

Remark 3.2. Note that we will only resolve the singularity at S, which is "generic" in the sense that it appears for all the values of the energy-momentum map. So the curve $\mathcal{C}$ can still be singular for some values of the energy-momentum map.

Finally we find the genus of $\mathcal{C}$ and $\mathcal{C}_m$ from the fact that the curves are hyperelliptic and from the degree of the polynomials in (7) and (9). We obtain $g(\mathcal{C}) = n - 1$ for normalized spectral curve $\mathcal{C}$ and the arithmetic genus $g_a(\mathcal{C}_m) = n$ for the singular curve $\mathcal{C}_m$.

3.2. Isospectral manifold of matrix polynomials

We have seen that the Neumann system satisfies Lax equation and that the spectral curve depends only on the first integrals and that all the first integrals are encoded in the spectral curve. The level sets of (complexified) energy momentum map F_{EM} lie in the set of matrices $L(\lambda)$ with fixed spectral curve $\mathcal{C}_m$. It is therefore essential to describe the set of matrix polynomials with a given spectral curve.

Let $P(\lambda, \mu)$ be *a spectral polynomial* for some matrix polynomial $L(\lambda) \in \tilde{\mathfrak{gl}}(r, \mathbb{C}))$. Denote with $\mathcal{M}_P$ the subset of all the elements of $\tilde{\mathfrak{gl}}(r, \mathbb{C})$ with the same characteristic polynomial P (this also fixes the spectral curve $\mathcal{C}$)

$$\mathcal{M}_P = \{L(\lambda); \quad \det(L(\lambda) - \mu) = P(\lambda, \mu)\}.$$

All $L(\lambda) \in \mathcal{M}_P$ have the same spectral curve $\mathcal{C}$, which is defined by $P(\lambda, \mu) = 0$.

While the characteristic polynomial and thus the eigenvalues of $L(\lambda)$ are fixed by the flow, the eigenvectors and eigenspaces change. Let us define a map

$$\xi_{L(\lambda)} : \mathcal{C} \to \mathbb{P}^{r-1},$$

such that $\xi_{L(\lambda)}((\lambda, \mu))$ is one dimensional eigenspace of the matrix $L(\lambda)$ with respect to the eigenvalue μ. If the spectral curve is smooth, than by proposition 3.1 all the eigenspaces of $L(\lambda)$ for any λ are onedimensional and the map ξ is well defined. The map $\xi_{L(\lambda)}$ defines a line bundle on $\mathcal{C}$ and its dual is called *eigenvector line bundle* or shorter *eigenline bundle.* We will denote eigenline bundle by $\mathcal{L}_{L(\lambda)}$. By construction, the eigenline bundle is a subbundle of the trivial bundle $\mathcal{C} \times \mathbb{C}^r$. One can see by using Riemann-Roch-Grothendick theorem that the degree (Chern class) d of the eigenline bundle $\mathcal{L}_{L(x)}$ equals $g + r - 1$ where g is the genus of the spectral curve $\mathcal{C}$.

The only condition for $\xi_{L(\lambda)}$ to be defined is that the eigenspaces of $L(\lambda)$ are one dimensional for all but finite number of points on $\mathcal{C}$. If the spectral curve is singular, than the map $\xi_{L(\lambda)}$ can be defined on the set of points $\mathcal{C}_m - N \subset \mathcal{C}_m$ where the matrix $L(\lambda)$ has one dimensional eigenspace. If the set N is finite, the map $\xi_{L(\lambda)}$ can be extended as a holomorphic map and eigenline bundle $\mathcal{L}_{L(x)}$ can be defined on the normalization $\mathcal{C}$ of the spectral curve. Note that by the proposition 3.1 the set N is a subset of singular locus of the spectral curve $\mathcal{C}_m$.

One can define the *eigenbundle map*

$$\begin{aligned} e : \mathcal{M}_P &\to Pic^d(\mathcal{C}) \\ L(\lambda) &\mapsto [\mathcal{L}_{L(\lambda)}], \end{aligned}$$

from $\mathcal{M}_P$ to the Picard group $Pic(\mathcal{C})$ of isomorphism classes of line bundles on the normalized spectral curve $\mathcal{C}$. The subset $Pic^d(\mathcal{C})$ consists of classes of line bundles with given Chern class d. The set $Pic^d(\mathcal{C})$ is a copy of the zero degree Picard subgroup $Pic^0(\mathcal{C})$, which is in turn isomorphic via Abel-Jacobi map to the Jacobian $Jac(\mathcal{C})$ of $\mathcal{C}$. The map e assigns to each matrix polynomial $L(\lambda)$ the isomorphism class of its eigenline bundle and thus encodes the flow of Lax equation. The map e is not surjective since eigenline bundles cannot lie in the special divisor Θ on the Jacobian. The map e is neither injective since the class of eigenline bundle defines the matrix polynomial only up to conjugation by the gauge group $\mathbb{P}Gl(r, \mathbb{C})$. The space we have to consider is the quotient space $\mathcal{M}_P/\mathbb{P}Gl(r, \mathbb{C})$ and it was shown in [5] that if the spectral curve $\mathcal{C}$ is smooth, the space $\mathcal{M}_P/\mathbb{P}Gl(r, \mathbb{C})$ is isomorphic as an algebraic manifold to the Zariski open subset $Jac(\mathcal{C}) - \Theta$ of the Jacobian of the spectral curve $\mathcal{C}$. The isomorphism is given by the eigenbundle map e.

As the leading coefficient in $L(\lambda)$ is preserved by the flow, we will use the closed subset of $\mathcal{M}_P$ of matrix polynomials with fixed leading coefficient. Let $L(\lambda) = A\lambda^l + A_{l-1}\lambda^{l-1} + \ldots + A_0$ and let A be fixed. We denote

$$\mathcal{M}_P^A = \{L(\lambda) \in \mathcal{M}_P; \quad \lim_{\lambda \to \infty} L(\lambda)/\lambda^l = A\}.$$

The action of the gauge group $\mathbb{P}Gl(r, \mathbb{C})$ on $\mathcal{M}_P$ reduces to the action of the stabilizer subgroup $\mathbb{P}G_A < \mathbb{P}Gl(r, \mathbb{C})$ of A. The quotient $\mathcal{M}_P^A$ by $\mathbb{P}G_A$ is again isomorphic to the Jacobian

$$\mathcal{M}_P^A/\mathbb{P}G_A \simeq Jac(\mathcal{C}) - \Theta.$$

Remark 3.3. If the spectral curve $\mathcal{C}$ is smooth at infinity, the matrix A has to be *regular* by proposition 3.1. As a consequence, the stabilizer group of A is the product

$\mathbb{P}G_A = (\mathbb{C}^*)^{s-1} \times \mathbb{C}^{r-s}$ where s is the number of distinct eigenvalues of A. If A is not regular, the dimension of the stabilizer group $\mathbb{P}G_A$ is larger. In our case, when $A = \mathrm{diag}(a_1, \ldots, a_n, a_n)$ the group $\mathbb{P}G_A$ is the product

$$(\mathbb{C}^*)^{n-2} \times Gl(2, \mathbb{C}).$$

Let us assume for one moment that the matrix A is regular. We have seen before that the level set $\mathcal{M}_P^A$ is an extension of $Jac(\mathcal{C}) - \Theta$ by an Abelian algebraic group $\mathbb{P}G_A$. As it happens, the generalized Jacobian is also defined as an extension by Abelian algebraic group and it was shown in [13] that

$$\mathcal{M}_P^A \simeq Jac(\mathcal{C}_m) - \Theta$$

for a suitable choice of modulus m. The chosen modulus is the effective divisor consisting of infinite points P_i on the spectral curve

$$m = \sum_{P_i \in \lambda^{-1}(\infty)} m_i P_i,$$

where the coefficients m_i are multiplicities of the eigenvalues $\mu(P_i)$ of $A = L(\infty)$. The isomorphism $\mathcal{M}_P^A \to Jac(\mathcal{C}_m) - \Theta$ is given by the eigenbundle map e_m to the generalized Jacobian. Let $S = |m|$ be the set of points in $\mathcal{C}$ that are mapped to the singular point in $\mathcal{C}_m$. The line bundle on the singular curve $\mathcal{C}_m$ is given by a divisor on $\mathcal{C}$ that does not intersect the singular set S. To give a line bundle on $\mathcal{C}_m$ is thus enough to give a section of a line bundle on $\mathcal{C}$ that has no zeros or poles in S. We can then extend the map $e : \mathcal{M}_P \to Pic^d(\mathcal{C})$ to a map

$$e_m : \mathcal{M}_P \to Pic^d(\mathcal{C}_m)$$

by choosing a section of $e(L(\lambda))$ uniformly on $\mathcal{M}_P$ that does not have any zeros or poles in the infinite point. This can be done by appropriate normalization. It was shown in [13, 14] that e_m gives an isomorphism from the isospectral space $\mathcal{M}_P^A$ to the Zariski open subset $Jac(\mathcal{C}_m) - \Theta$ of the generalized Jacobian of the singular spectral curve, given by modulus m.

Remark 3.4. The above results give orientation about the expected number of degrees of freedom of isospectral flows. We see that the upper limit is the arithmetic genus of the singularization of the spectral curve, which depends on the degree l and the rank r of $L(\lambda)$.

In the case of confluent Neumann system, the matrix A is not regular and corresponding spectral curve is singular. We will show later that the Lax flow also preserves part of the "derivative" of $L(\lambda)$ at the singular point. We will restrict the space $\mathcal{M}_P^A$ further by fixing a specific block of the lower term A_{l-1}. The resulting isospectral set will again be isomorphic to the open subset of the generalized Jacobian [14]. Note that we have to assume regularity of the fixed block of the lower term A_{l-1} in order to have the isomorphism.

3.3. Proof of the integrability

In order to prove the Liouville integrability of the confluent Neumann system, we have to show that its first integrals $\{F_1, \ldots, F_{n-1}, K\}$ Poisson commute and that they are functionally independent. We already know that the integrals commute from section 2. We also know that the level sets of the first integrals - the Arnold-Liouville tori - lie in the isospectral manifold $\mathcal{M}_P^A$. We will use the eigenbundle map e_m to map Arnold-Liouville tori to the real part of the generalized Jacobian of the singular spectral curve

$\mathcal{C}_m$ and show that this map is a $(\mathbb{Z}_2)^{n-1}$ covering. This will prove complete algebraic integrability of confluent Neumann system and independence of the first integrals will follow.

Let E_n be the eigenspace of the double eigenvalue a_n, which is spanned by the unit vectors e_n and e_{n+1}. The behavior of $L(\lambda)|_{E_n}$ near infinity is given by the $n, n+1$ block F of the matrix $q \wedge p$, which is in fact conserved by the isospectral flow. The block F depends only on the angular momentum $K = q_n p_{n+1} - q_{n+1} p_n$ and is a regular matrix

$$F = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}$$

of rank 2 if $K \neq 0$. The isospectral flow given by (6) conserves both matrix A and the block F, therefore it is reasonable to consider isospectral manifold of all matrix polynomials with this data fixed

$$\mathcal{M}_P^{A,F} = \{L(\lambda) \in \mathcal{M}_P; \quad L(\infty) = A,\ \mathrm{pr}_{E_n} \circ L'(\infty)|_{E_n} = F\}$$

where pr_{E_n} is a projection to E_n, and the values of $L(\lambda)$ and $L'(\lambda)$ at infinity are defined by the limits $L(\infty) := \lim_{\lambda\to\infty}(\lambda^{-2}L(\lambda))$ and $L'(\infty) := \lim_{\lambda\to\infty} \mathrm{d}(\lambda^{-2}L(\lambda))/\mathrm{d}(\lambda^{-1})$. On $\mathcal{M}_P^{A,F}$ acts the subgroup $\mathbb{P}G_{A,F} < \mathbb{P}G_A$ of matrices that stabilize F as well. We can write

$$\mathbb{P}G_{A,F} \simeq (\mathbb{C}^*)^{n-1} \times G_F$$

where G_F is the stabilizer subgroup of F in $Gl(2,\mathbb{C})$. The group $G_F \simeq \mathbb{C}^*$ consists of matrices

$$r \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix},$$

where $r \in (0,\infty)$ and $\varphi \in [0, 2\pi)$. We will describe the isospectral manifold $\mathcal{M}_P^{A,F}$ as an open subset of the generalized Jacobian of singularized spectral curve.

Theorem 3.5. *Let $f \in \mathbb{C}^n$ be the value of F_{EM} such that the normalized spectral curve $\mathcal{C}$ is smooth and $K \neq 0$. Let denote by $\mathbb{T}_A = (\mathbb{C}^*)^{n-1}$ the subgroup of $\mathbb{P}G_A$ of diagonal matrices $G = [g_{i,j}]$ with $g_{n,n} = g_{n+1,n+1}$.*

(i) The complex level set $F_{EM}^{-1}(f)$ is a covering of the isospectral manifold $\mathcal{M}_P^{A,F}/\mathbb{T}_A$. The fiber of the covering is the same as the orbit of the group $(\mathbb{Z}_2)^{n-1}$ generated by reflections $q_i \mapsto -q_i$ for $1 \leq i \leq n-1$.

(ii) The isospectral manifold $\mathcal{M}_P^{A,F}/\mathbb{T}_A$ is isomorphic to the open subset of the generalized Jacobian of the singular spectral curve $\mathcal{C}_{m'}$ given as a singularization of smooth spectral curve by modulus $m = (\infty, \mathrm{i}K) + (\infty, -\mathrm{i}K)$.

(iii) The flow of K generates the fib re $\mathbb{C}^*$ of the extension $\mathbb{C}^* \to Jac(\mathcal{C}_m) \to Jac(\mathcal{C})$.

(iv) The flow of H on the generalized Jacobian $Jac(\mathcal{C}_m)$ is linear.

Proof. If $K \neq 0$ then F is a regular matrix with 2 different eigenvalues $\pm\mathrm{i}K$. By the result in [14] the isospectral manifold $\mathcal{M}_P^{A,F}$ is isomorphic as an algebraic manifold to the Zariski open subset $J(\mathcal{C}_m) - \Theta$ of the generalized Jacobian of the singularized spectral curve, given by spectral polynomial P and modulus $m' = \lambda^{-1}(\infty) =$

$\sum_{i=1}^{n-1}(\infty, a_i) + (\infty, \mathrm{i}K) + (\infty, -\mathrm{i}K)$. The same theorem asserts that the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{P}G_{A,F} \arrow[r] \arrow[d] & \mathcal{M}_P^{A,F} \arrow[r] \arrow[d, "e_{m'}"'] & \mathcal{M}_P^{A,F}/\mathbb{P}G_{A,F} \arrow[r] \arrow[d, "e"'] & 0 \\ 0 \arrow[r] & (\mathbb{C}^*)^n \arrow[r] & Jac(\mathcal{C}_{m'}) - \Theta \arrow[r] & Jac(\mathcal{C}) - \Theta \arrow[r] & 0 \end{tikzcd}$$

If we singularize $\mathcal{C}$ only by modulus $m = (\infty, \mathrm{i}K) + (\infty, -\mathrm{i}K)$, we can insert $e_m : \mathcal{M}_P^{A,F}/(\mathbb{C}^*)^{n-1} \to Jac(\mathcal{C}_m) - \Theta$ into above diagram

$$\begin{tikzcd} & G_F \arrow[d] & \\ \mathcal{M}_P^{A,F} \arrow[r] \arrow[d, "e_{m'}"'] & \mathcal{M}_P^{A,F}/\mathbb{T}_A \arrow[r] \arrow[d, "e_m"'] & \mathcal{M}_P^{A,F}/\mathbb{P}G_{A,F} \arrow[d, "e"'] \\ Jac(\mathcal{C}_{m'}) - \Theta \arrow[r] & Jac(\mathcal{C}_m) - \Theta \arrow[r] & Jac(\mathcal{C}) - \Theta \\ & \mathbb{C}^* \arrow[u] & \end{tikzcd}$$

We only have to prove that the fiber $F_{EM}^{-1}(f)$ consisting of matrices of the form (5) is a covering of the quotient $\mathcal{M}_P^{A,F}/\mathbb{T}_A$. By using the Lax matrix (5) we gave a parametrization

$$J^A : (q,p) \mapsto L(\lambda) = A\lambda^2 + q \wedge p\ \lambda - q \otimes q$$

of the quotient $\mathcal{M}_P^{A,F}/\mathbb{T}_A$ by $(q,p) \in F_{EM}^{-1}(f) \subset (T^*S)^{\mathbb{C}}$. First note that the map $J^A : (T^*S^n)^{\mathbb{C}} \to \mathcal{M}_P^{A,F}$ is an immersion. We will show that any orbit of $\mathbb{T}_A$ intersects the image of J^A only in finite number of points. To explain how $\mathbb{P}G_{A,F}$ acts on the Lax matrix (5), note that an element $g \in \mathbb{P}G_{A,F}$ acts on a tensor product $x \otimes y$ of $x, y \in \mathbb{C}^{n+1}$

$$g : x \otimes y \mapsto (gx) \otimes ((g^{-1})^T y)$$

by multiplying the first factor with g and the second with $(g^{-1})^T$. The subgroup of $\mathbb{P}G_{A,F}$, for which the generic orbit lies in the image of J^A, is given by orthogonal matrices

$$O(n,\mathbb{C}) \cap \mathbb{P}G_{A,F} \simeq (\mathbb{Z}_2)^{n-2} \times G_F \cap O(2,\mathbb{C}).$$

There are special points in the image of J^A that have a large isotropy group (take for example $q_i = \delta_{ij}$ and $p_i = \delta_{ik}$, $k \neq j$, where the isotropy is $(\mathbb{C}^*)^{n-2}$). But the intersection of any orbit with the image of J^A coincides with the orbit of $(\mathbb{Z}_2)^{n-2} \times G_F \cap O(2,\mathbb{C})$. If we take the torus $\mathbb{T}_A < \mathbb{P}G_{A,F}$ consisting of diagonal matrices with $g_{n,n} = g_{n+1,n+1}$, so that $\mathbb{T}_A \cap G_F = \{Id\}$, the orbits of $\mathbb{T}_A$ will intersect image of J^A only in the orbit of the finite subgroup $(\mathbb{Z}_2)^{n-2}$. We have proved that the level set of Lax matrices $L(\lambda) = J^A(q,p)$ with fixed characteristic polynomial P is an immersed submanifold in $\mathcal{M}_P^{A,F}$ that intersects the orbits of torus $\mathbb{T}_A \simeq (\mathbb{C}^*)^{n-1}$ in only finite number of points and is thus a covering of the quotient $\mathcal{M}_P^{A,F}/\mathbb{T}_A$.

The group G_F is the complexification of the group of rotations in q_n, q_{n+1} plane and is generated by the Hamiltonian vector field of K. This proves the assertion (iii).

To prove the assertion (iv), note that the matrix polynomial $M(\lambda)$ in Lax equation (6) is given as a polynomial part $R(\lambda, L(\lambda))_+$ for a polynomial $R(z,w) = zw$. It is well known that such isospectral flows are mapped by e_m to linear flows on the Jacobian $Jac(\mathcal{C}_m)$ (see [4] for reference). □

Taking into account the real structure on $\mathcal{C}_m$, the Arnold-Liouville tori can be described as a real part of the generalized Jacobian.

Theorem 3.6. *For $K \neq 0$ and $\mathcal{C}$ smooth, the Arnold-Liouville tori are $(\mathbb{Z}_2)^{n-2}$ coverings of the real part of the generalized Jacobian. The rotations generated by K are precisely the rotations of the fiber S^1 in the fibration $S^1 \to Jac(\mathcal{C}_m)^{\mathbb{R}} \to Jac(\mathcal{C})^{\mathbb{R}}$, which is the real part of the fibration $\mathbb{C}^* \to Jac(\mathcal{C}_m) \to Jac(\mathcal{C})$.*

Above theorem gives us an algebraic way to describe symplectic quotient $T^*S^n//_k S^1$. Algebraically T^*S^n is a covering of the relative generalized Jacobian, which is a disjoint union

$$\tilde{J}ac^{\mathbb{R}} = \cup_{\mathcal{C}_m} Jac^{\mathbb{R}}(\mathcal{C}_m)$$

over the space of curves $\mathcal{C}_m$ corresponding to the real values of energy momentum map. The symplectic quotient of $\tilde{J}ac^{\mathbb{R}}//_k S^1$ is then the relative Jacobian over the space of normalized spectral curves with fixed value of K

$$\cup_{\mathcal{C}_m;\, K=k} Jac^{\mathbb{R}}(\mathcal{C}).$$

Corollary 3.7. *The complex level set of $(F_1, \dots, F_{n-1})$ of the symplectic quotient of the confluent Neumann system is a $(\mathbb{Z}_2)^{n-2}$ covering of the quotient $\mathcal{M}_P^{A,F}/\mathbb{P}G_{A,F}$. The manifold $\mathcal{M}_P^{A,F}/\mathbb{P}G_{A,F}$ is isomorphic to the open subset $Jac(\mathcal{C}) - \Theta$ of the Jacobian of the normalized spectral curve.*

This result agrees perfectly with the results obtained previously for the Rosochatius system [11].

Proof of the theorem 3.6. Note that the eigenvalues of F are $\pm \mathrm{i}K$. Note also that the value of μ at the points $P_\pm$ equals to the eigenvalues of F, so $P_\pm = (\infty, \pm\mathrm{i}K)$. On $\mathcal{C}$ there is a natural real structure J induced by the conjugation on $(\lambda, \mu) \in \mathbb{C}^2$. The points $P_\pm$ that are glued in the singular point form a conjugate pair $P_\pm = JP_\mp$. If we follow the argument in [17] we can find the real structure of the fiber $\mathbb{C}^*$ in the extension $\mathbb{C}^* \to Jac(\mathcal{C}_m) \to Jac(\mathcal{C})$. Note that $Pic(\mathcal{C})$ is defined as the space of all divisors modulo divisors of meromorphic functions on $\mathcal{C}$, whereas $Pic(\mathcal{C}_m)$ is given by the divisors on $\mathcal{C}$ that avoid $P_\pm$ modulo meromorphic functions on $\mathcal{C}_m$. So the fiber $\mathbb{C}^*$ is given by meromorphic functions on $\mathcal{C}$ modulo meromorphic functions on $\mathcal{C}_m$. Since we obtained $\mathcal{C}_m$ by gluing two points $P_\pm$, a function f on $\mathcal{C}$ defines a function on $\mathcal{C}_m$ if $f(P_+) = f(P_-)$ or equivalently $f(P_+)/f(P_-) = 1$. For a divisor of any function f on $\mathcal{C}$, the number $z = f(P_+)/f(P_-) \in \mathbb{C}^*$ determines its class in the Picard group $Pic(\mathcal{C}_m)$. So if $P_\pm = JP_\mp$, then the real structure on the fiber $\mathbb{C}^*$ is given by the map

$$z = \frac{f(P_+)}{f(P_-)} \to \frac{\overline{f(JP_+)}}{\overline{f(JP_-)}} = \frac{\overline{f(P_-)}}{\overline{f(P_+)}} = \frac{1}{\bar{z}}$$

and the real part of $\mathbb{C}^*$ is the unit circle S^1 given by $z\bar{z} = 1$. In contrast, when the singular points are real $P_\pm = JP_\pm$, the real structure on $\mathbb{C}^*$ is given by the conjugation and the real part of $\mathbb{C}^*$ is $\mathbb{R}^*$. □

3.4. Bifurcation diagram

We can use the normalized spectral curve $\mathcal{C}$ to describe the singular locus of energy momentum map.

Proposition 3.8. *The vector $(f_1, \ldots, f_{n-1}, k) \in \mathbb{R}^n$ is a regular value of the real momentum map*

$$F_{EM} = (F_1, \ldots, F_{n-1}, K)$$

if and only if the normalized spectral curve $\mathcal{C}$ is smooth. The singular locus of the map F_{EM} consist of

- *hyperplanes $F_i = 0$*
- *zero level set of the discriminant of $Q(x)$ from (8)*
- *the codimension 2 hyperplane defined by $K = 0$ and $H = \frac{1}{2}a_i$*

Proof. The if part follows directly from theorem 3.5. If $\mathcal{C}$ is smooth and $K \neq 0$, then the level set of F_{EM} is locally isomorphic to the real part of $Jac(\mathcal{C}_m) - \Theta$, which is of dimension n. Hence the rank of the differential of F_{EM} is also n. If $K = 0$ and $\mathcal{C}$ is smooth, the level set of $(F_1, \ldots, F_{n-1})$ is locally isomorphic to the real part of the isospectral manifold $\mathcal{M}_P^{A,F}/\mathbb{P}G_{A,F}$, which is in turn isomorphic to the real part of $Jac(\mathcal{C}) - \Theta$. Since the dimension of $Jac(\mathcal{C})$ is $n-1$, the rank of the differential of $(F_1, \ldots, F_{n-1})$ is also $n-1$. The integrals F_i are invariant to the rotations generated by K and therefore their Hamiltonian vector fields $X_{F_1}, \ldots, X_{F_{n-1}}$ are independent from X_K. We are left to show that if $X_K = 0$ the normalized curve $\mathcal{C}$ is not smooth. It is easy to see that the case $X_K = 0$ appears only if $X_H = 0$ but then the rank of $X_{F_1}, \ldots, X_{F_{n-1}}$ is not full and the curve $\mathcal{C}$ has to be singular.

To prove the only if part let us consider case by case the components of the singular locus. The curve $\mathcal{C}$ is singular if and only if the polynomial $\prod(a_i - x)Q(x)$ has a double root. This can happen in two cases

(i) a_i is a zero of $Q(x)$, this happens when $F_i = 0$

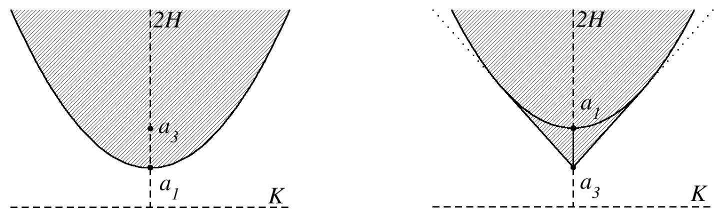
(ii) $Q(x)$ has double zero, this happens if the discriminant of Q is zero.

The hyperplanes $F_i = 0$ are singular, since for the points (q, p) with $q_i = p_i = 0$ the differential $\mathrm{d}F_i = 0$. The proof that the discriminant of Q is singular can be found in [6] and I will omit it here, because it is very specific and beyond the scope of this article. □

Remark 3.9. For values of $a_n > a_j$ for some $1 \leq j < n$ the bifurcation diagram has a singular "thread" of focus-focus singularities defined by values $K = 0$ and $H = \frac{1}{2}a_j$. This would suggest the presence of nontrivial monodromy. Indeed for two degrees of freedom, the singular level set corresponding to the isolated singular value is a union of two spheres with two pairs of points identified. By the general result in [23] it follows that the monodromy is nontrivial and equals

$$\begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix} \tag{10}$$

with $\varepsilon = 2$ being the number of spheres in the singular level set. This can also be checked by direct calculation.



(a) $a_1 > a_2$ (b) $a_2 > a_1$

Figure 1: Image of the energy-momentum map $(K, 2H)$ for confluent Neumann system with potential matrix $A = \mathrm{diag}(a_1, a_2, a_2)$. Regular values lie in the shadowed area while the solid curves contain singular values. The points $(0, a_1)$ and $(0, a_2)$ are the images of fixed points.

The image of the energy momentum map $(K, 2H)$ for confluent Neumann system with two degrees of freedom is depicted in figure 1. Regular values lie in the shadowed area while the solid curves contain singular values. The points $(0, a_1)$ and $(0, a_2)$ are the images of fixed points. Note that when passing from $a_2 > a_1$ to $a_1 > a_2$, the pair of lines becomes imaginary and only their intersection - isolated focus-focus point $(0, a_1)$ - remains real.

3.5. Note on superintegrability

An integrable system with Hamiltonian H and a commuting set of first integrals $H = F_1, \ldots, F_n$ is superintegrable, if there exist additional first integrals that Poisson commute with H but not with all F_i (see [24] for reference).

One would expect that confluenting eigenvalues of the potential would result in additional symmetries and superintegrability. This is the case if the potential has $k > 2$ identical eigenvalues and the system is invariant to the action of non commutative group $SO(k)$, which gives rise to additional first integrals that do not Poisson commute with each other.

However if only two of the eigenvalues coincide, the confluent Neumann system is not superintegrable. The proof of this conjecture is beyond the scope of this article, but we provide the argument in two degrees of freedom. In that case superintegrability implies that the flow of H is periodic. If we write $H = p_1 I_1 + p_2 I_2$ with the action integrals $I_1 = \frac{K}{2\pi}$ and I_2, the flow of H is periodic if and only if the quotient $\frac{p_1}{p_2}$ is rational. However the presence of nontrivial monodromy (10) for $a_1 > a_2$ implies, that the quotient $\frac{p_1}{p_2}$ increases by $\varepsilon = 2$ if we make one turn around the singular point $(0, a_1)$ in the image of $(K, 2H)$. Therefore the quotient $\frac{p_1}{p_2}$ has to be irrational for generic values of H and K.

3.6. Note on the case $K = 0$

We have seen in the previous section that the case $K = 0$ is significantly different from the generic case $K \neq 0$. The problem lies in the following observation. The space of hyperelliptic curves that appear in the description of Neumann system is parametrized by K^2 and not by K. As a consequence, the relative generalized Jacobian is degenerated for $K = 0$. The phase space of complexified Neumann system is therefore "folded" into relative Jacobian by the map $K \to K^2$. The map between original phase space and its image in relative Jacobian is singular at $K = 0$. It is therefore illusory to expect that we can describe the whole phase space including the fiber $K = 0$ by algebro-geometric methods. Different approach has to be considered that would study the "fold" given by K^2 in more detail. This is to be covered in our future work.

4. Conclusions and discussion

We have proved the algebraic integrability of the confluent Neumann system by proving the theorem 3.5, which describes Arnold-Liouville tori in terms of the generalized Jacobians of singular spectral curve. We performed the reduction of the rotational symmetry and established a firm relationship between symplectic reduction and desingularization of the spectral curve (corollary 3.7). Most of our results very likely generalize to any Moser system arising from the rank 2 perturbations of a fixed matrix with a double eigenvalue. From our work and previous examples [17, 16] it appears that there generally is a relation between the rotational symmetry and singularities of spectral curves. We have exposed this relationship explicitly in our case and have seen that the reduction of S^1 symmetry reveals itself in the algebraic description as a reduction from generalized Jacobian of the singular spectral curve to the "ordinary" Jacobian of the normalized spectral curve. One can say that the desingularization of the spectral curve corresponds to the symplectic quotient. Unfortunately, this relation is not a general phenomenon as we can see when considering the case $K = 0$. One can speculate that the appearance of the global action of a compact group is related to the presence of a "generic" singularity but there is no general proof yet. Note that the generic Neumann system has no symmetries given by a compact group.

The singularities of the spectral curve appeared in two different roles in our study. The singularity that is a consequence of the confluency is "generic" in that it appears uniformly for all values of the energy-momentum map. The rotational symmetry shows as the extension by complexified group of rotations $\mathbb{C}^*$ defining the generalized Jacobian that appears globally. The "sporadic" singularities, which correspond to the singular values of the energy momentum map (see proposition 3.8) are strictly a local phenomenon. In those cases the extension by $\mathbb{C}^*$ and the resulting rotational symmetry does not extend globally. Algebraically speaking both singularities are the same, but the "generic" singularity appears globally and thus give rise to a rotational symmetry. Sporadic singularities on the other hand appear when the level sets of energy momentum map are singular (orbits of lower dimension, heteroclinic and homoclinic orbits). It would be interesting to describe the isospectral sets of the singular spectral curves. Note that when we introduced generic singularity we made sure that we used the subset of the singular isospectral set, consisting of regular Lax matrices. In the study of sporadic singularities, non regular part of the isospectral set should not be avoided. It is our conjecture that the singular isospectral sets that induce homoclinic or heteroclinic orbits should pose an obstruction to the existence of global action of compact groups.

In a somewhat more ambitious and speculative vein, one could study the relationship between symmetries of certain PDE's and generic singularities in appropriate spectral curves of infinite genus. Maxwell-Bloch equation for example can be viewed as a chain of confluent Neumann systems[25, 26], whose symmetries indeed reflect in a symmetry of the whole Maxwell-Bloch system [27]. The description of Maxwell-Bloch system with generalized Jacobians of singular spectral curves of infinite genus should be to some extent analogous to our results.

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